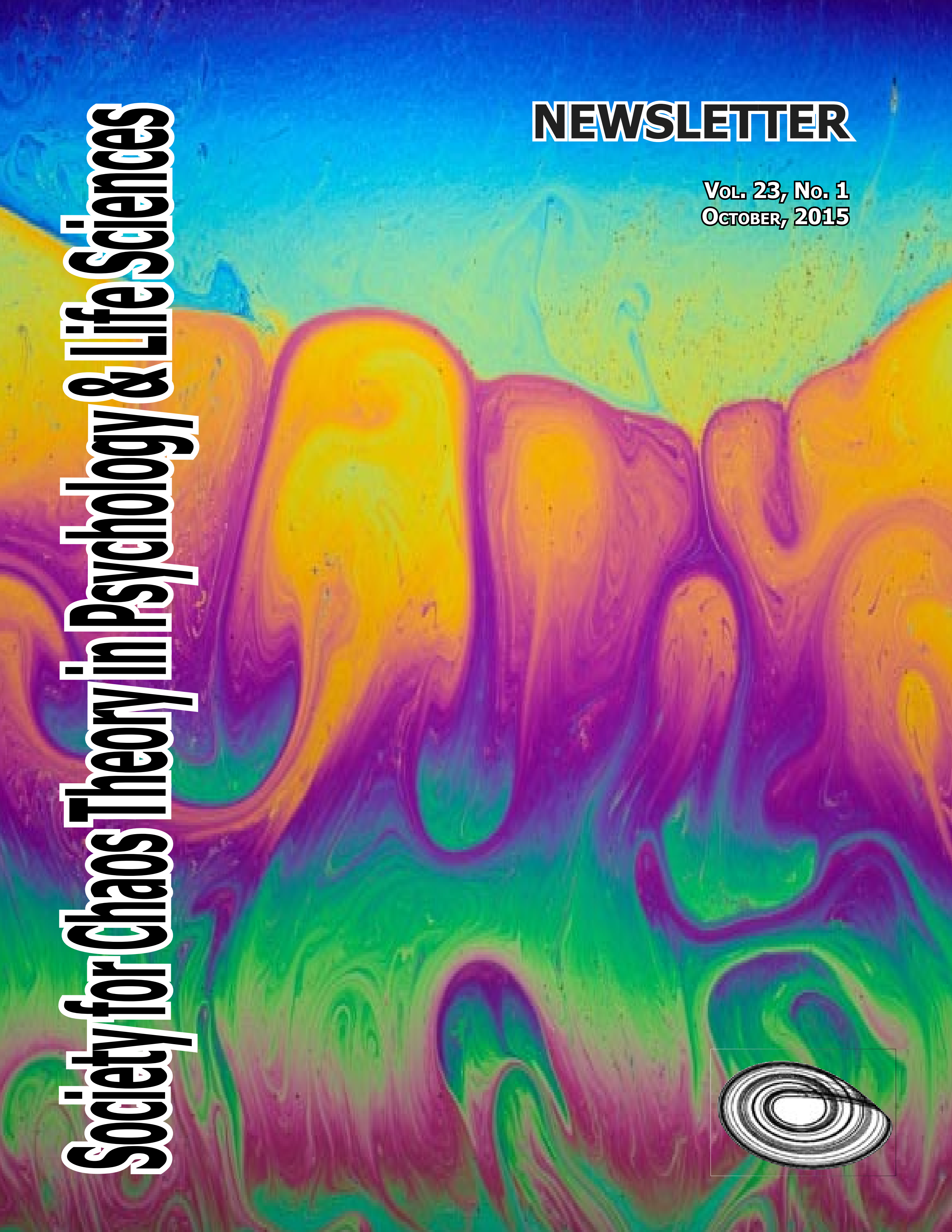

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Feature Article

QUATERNIONS, CHIRALITY, EXCHANGE INTERACTIONS: A NEW TOOL FOR NEUROSCIENCE?

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Martin A. Hay is an independent researcher, inventor, pharmaceutical patent attorney, and former chemist with an interest in chirality, quaternions and neuroscience who has invented a chiral system for coding information about economic relationships (exchange, taxation and voting).
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Introduction

Ronald Anderson and Ghirish C. Joshi, in a 1993 philosophy of science essay, "Quaternions and the heuristic role of mathematical structures in physics," describe quaternions as a model and tool (Anderson and Joshi, 1993):

> One of the most important ways development takes place in mathematics is via a process of generalization. On the basis of a recent characterization of this process we propose a principle that generalizations of mathematical structures that are already part of successful theories serve as good guides for the development of new physical theories. The principle is a more formal presentation and extension of a position stated earlier in this century by Dirac. *Quaternions form an excellent example of such a generalization* and we consider a number of ways in which their use in physical theories illustrates this principle. [*Note: italics emphasis is ours]*

Quaternion History and Characteristics

Quaternions were invented by William Hamilton in 1843. During the preceding decade, a number of mathematicians, including the logician Augustus De Morgan and the mathematician/computer algorithm-designer Ada Lovelace (Qayoom, 2009), wished to extend the well-situated 2D complex numbers to higher dimensions, aiming at 3 dimensions, wishing to establish a system for describing and analyzing operations on objects in 3D spaces. Hamilton eventually realized that such a 3D operational tool itself lay in 4D space, not 3D space. Quaternion elements are based on three distinct imaginary numbers, usually referred to as i, j, and k.
Quaternions contain two parts – a scalar part (an amount) and a vector part (a directed line segment).
Quaternions as operators (that is, functional elements) produce rotations and scaling. Below is a brief historical chronology of quaternions (Crowe, 1994; Cohen, 2007):

Quaternions were very prominent in the 19th Century in the life of many universities in the U.S. and Europe. At Harvard, because of the influence of a very early developer of quaternions, Benjamin Peirce (father of philosopher and logician Charles Sanders Peirce), all graduate math courses were taught with the use of quaternion tools.

Quaternions were controversial, to some because of their complicated and unstandardized notation and conventions and to others because of their unreal 4D nature. But they were praised as an algebraic/geometric system thinking tool by Clerk Maxwell.

By 1910, quaternions were largely replaced in universities by newer vector tools developed independently by Gibbs and Heaviside.

Despite this trend, they were skillfully used in the early 20th century in quantum physics (Pauli 2x2 spin matrices) and by the famous cognitive development researcher, Jean Piaget (Piaget et al, 1977; Piaget and Inhelder, 2001). They were also strongly supported in philosophy of science by E.T. Whittaker (Whittaker 1903, 1904).

In the period 1985 to 2015, quaternions were re-employed in scientific and technical research, often in computer-driven environments. Motives included better accuracy and reliability, processing speed, and parameter clarity. Quaternions have now been used in many research areas. The most prominent or promising include: Aerospace guidance and control, Computer graphics and animation, Signal/image processing, Bio-logging (free-range animal and human dynamic orientation analysis and wireless transmission),Cognition and music processing – nine research papers involving *4D models over real numbers* (representable by quaternions) were cited from the literature in an investigation of quaternion applications to cognition (Klitzner, 2015b); also, Terry Marks-Tarlow explored the connection between quaternions and fractals, and between quaternions and generalized memory of location (Marks-Tarlow, 2004).

Yet most science researchers have not yet heard of quaternions – even their name. This is slowly changing.

Quaternions and Chirality

The fundamental unit of information in biology is a chiral tetrahedral molecule, which is itself and not its enantiomer (3D mirror image). It is self-referential. For desired medical effects, drug molecules must fit hand in glove (outer space in inner space), which is why handedness matters – the mirror image won't fit. In a series of papers Salvatore Capozziello and Alessandra Lattanzi (Capozziello and Lattanzi, 2005, 2006) have shown that chiral tetrahedral molecules can be described mathematically as unitary quaternions. Martin is very grateful to these two authors for their help, which has guided the development of the work published in this paper. Chiral tetrahedral molecules are composed of four different atoms or chemical groups ordered as ligands bound to a central atom (carbon in living systems). The quaternion representation of the molecule conserves the order of the ligands as the molecule is rotated in 3D space and hence contains a memory of their spatial relationships to one another. The two enantiomers of a chiral tetrahedron can be nested in a cube, such that each of their four like ligands are arranged at opposed corners.

The symmetry of the cube is broken, because one enantiomer is privileged over the other. The cube thus contains two nested quaternions, each the 3D mirror image of the other. Each of the four ordered elements of the quaternion has a dual. In this paper we will show how these four ordered elements function as four ordered polarities in accordance with quaternion multiplication rules and how this property can be related geometrically to rotations and inversions of the cube in 3D space and to exchange interactions between quaternions.

Comparison of Conventional and Chiral Cube Quaternion Usage – Notation and Combining Rules

A quaternion is composed of four ordered elements: 1, i, j, and k, each of which can be preceded by a coefficient a, b, c and d as in a.1+b.i+c.j+d.k. The coefficients can be positive, negative or zero. Thus, for example, the quaternion j is 0.1+0.i+1.j+0.k. When two quaternions are multiplied together, the coefficients are multiplied and summed according to a formula.

In chiral cube quaternion usage each quaternion is composed of four ordered polarities: ±1, ±i, ±j, ±k. For example, the quaternion j is -1, +i, +j, -k. The coefficients can be positive or negative, not zero. When one quaternion is multiplied by another, each polarity is multiplied by the same quaternion. For example, i.j is i.-1+i.i+i.j+i.-k = -1-i +j+k which is k. The effect is that the quaternions transform in a unitary manner. Each element in the quaternion can thus be associated with a binary choice, such as in excitation or inhibition.

Connecting Inner Space and Outer Space

Bernd Schmeikal is a pioneer in the logical and mathematical foundations of cognition (Schmeikal 2015a, 2015b) – how the cognitive construction of inner space and outer space connects. He has imaginatively brought together the works of philosopher Immanuel Kant (inner knowledge), mathematician Louis Kauffman (iterant algebra) (Kauffman 1987a, 1987b), and cognitive theorist Arnold Trehub (the retinoid space) (Trehub 1994, 2007, 2013), using a system of *four ordered polarities* by constructing structured environments from co-tessarines and quaternions:

> We can assume that in the human brain, where space-time is mapped onto inner space by natural constitution, there is exchange of information in quaternion arrays. The whole arrangement of sentences is always bound to the algebra of space-time. It is so to say coupled with the light phenomena of our biological existence. To find the correct mathematical terms, it is necessary to realize that the indicational space of a cognitive re-entry form is a fourfold array.

The term "four ordered polarities" has been adopted by us as a generic description of the unitary quaternions following our contact with the work of Bernd Schmeikal, because he relates fundamental algebraic structures to four-membered polarity strings. (However, this is not to suggest that a correspondence between the two usages has been found, but to acknowledge that a comparison may reveal one.)

Biology and Consciousness

An issue of *Computing Now (IEEE publication) in 2013* was devoted to the convergence of biological processing and artificial intelligence processing. One contributor, Paul F.M.J. Verschure (Verschure, 2013), suggested that the focus of AI be understanding consciousness rather than intelligence:

After chasing the mirage of intelligence for the past 60 years, AI researchers haven't made significant progress in a system-level understanding of mind as evidenced by our current inability to engineer advanced human-compatible autonomous systems. Conversely, neuroscience is chasing the dream of big data and running the risk of losing sight of its goal to understand the brain by sacrificing hypotheses and theory. I propose that by moving from intelligence to consciousness, we can find a new and integrated science and engineering of body, brain, and mind that will not only allow us to realize advanced machines but also directly address the last great outstanding challenge faced by humanity: the nature of subjective experience.

What are the internal tools of subjectivity? How might they be derived from broader tools of control in nature? Quaternion mathematics and its expression in the chiral cube is a candidate of interest.

Significance of Quaternions to Biology

From a very early age, we were all taught to think in a three number system: mutually negating +1 and -1, and zero. We learned addition, then subtraction, and then abstracted this further to a concept of negative numbers and debt. Now, this three numbers system permeates all our models, even the quaternions, for which each of the four elements can be positive, negative or zero.

Biology does not work in this way. It is binary. A neuron can fire or not, and it can do so in response to the presence or to the absence of a stimulus. The coding of presence and absence are both active processes. A blind person cannot draw a distinction between a presence and an absence (at least consciously, though interestingly some blind people can do so unconsciously).

What is the difference between 1 and 0? Biology codes each of 1 and 0 actively. Now here is the critical jump to make. In biology, 1 and 0 are polar opposites. They are related as +1 to -1, *but in biology +1 and -1 do not mutually negate.* Each can signify a presence or an absence, so we could equally use 0 to represent positive as negative, as long as we are consistent with how we treat the two polar opposites.

Instead of trying to code things in terms of ordered pairs of 1s and 0s, you need to use an ordered pair of ordered pairs: four ordered polarities. When you do that, everything falls into place. Everything can be understood in terms of the way in which two sets of four ordered polarities line up. The switch from positive to negative and the switch from sine to cosine corresponds with the reversal of polarities. Moreover, you can understand this geometrically in relation to the rotation of a chiral cube. The system is controlled by chirality.

It appears that the brain works in waves, and we think of waves in terms of rotation about a circle relative to a reference point (0) on that circle. A sine wave moves between positive, zero and negative. Contemporary papers about brain oscillations may be getting in the way of embracing this different system of coding, because they are rooted in three number thinking (+1, -1, 0; sine waves). We are looking at a 4D system coded in four ordered oppositional pairs. It is all as simple as lining up four pennies in a row and flipping them over as between heads and tails whenever there is a state change/retuning.

With this system, you can have seemingly paradoxical states like "true, false" and "not true, not false", but they can disentangle into clean oppositional pairs like "true, not false" and "not true, false". Behind every clean oppositional pair are others ready to rotate into view as the train of thought progresses. This kind of complexity works well to model ambiguous and paradoxical mental states. Meanwhile, you can use pairs of quaternions to think yourself out of paradoxes, such as right and left. When right and left become indistinguishable, another oppositional pair comes to mind. A boundary is always something and nothing. Each collapse of one oppositional pair and opening of another at a boundary can be modelled as coupled rotations of two chiral cubes.

This system may be useful for modelling multistability in the brain, where boundary conditions change all the time, at the edge of criticality, as the same underlying neural networks are re-configured in different combinations over and over. It is because biology works in only one chirality that it can be self-referential: conscious of itself. At least two rotations separate the two members of an oppositional pair.

Social Exchange, Ownership, and Control

Ownership: Basically, unitary quaternions (which are also referred to as chiralkine numbers) can encode ownership relationships (Hay, 2015, 2012; Hay and Hay, 2012). An exchange interaction can code a change in ownership relationships.

The key to seeing this lies in recognising that the ligands of a quaternion (1, i, j and k) can be assigned meanings, for example as in me, you, mine and yours, bound to an object, as in a possession. Unitary quaternions can keep track of the relationships between the ligands through cycles of exchange.

Ownership is an antisymmetric relationship. We are conscious of it. It reverses polarity on exchange of identities. The relationships of two people to an object that one owns and the other does not is reversed when the identities of the two people are exchanged. The relationship of a third party to the object before and after the ownership has been transferred is symmetric with respect to the transfer.

Unitary quaternions can encode the relationships of each person to the object as its ownership is transferred from one to another. Things can change from one perspective, but stay the same from another. If you are on a quaternion rotation axis, nothing changes from your perspective.

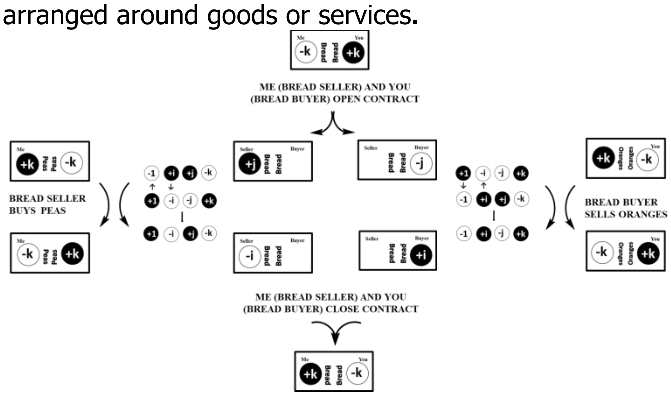
The perception of ownership is actually central to our very being. We own our lives and our bodies. They belong to us, not to anyone else. Other species also have a sense of self, and of possession of territories that they defend or respect.

Symmetry and Antisymmetry: In our homes that we share with our partners in a relationship, everything has its place. Beneath this order is something more. Our knives and forks go in the kitchen drawer, not in the neighbors'. The neighbors' food goes in their pantry, not ours. The locating of these objects in their places is symmetric with respect to the two of us. However, my wife's clothes go in her chest of drawers, not mine, and mine go in my chest of drawers, not hers. The locating of these objects in their places is antisymmetric with respect to the two of us.

This information is encoded in quaternions as unitary wholes. It can also be recovered in terms of symmetric and antisymmetric relationships as between the two of us and the neighbors. To the neighbors, the clothes are ours, not theirs – an antisymmetric relationship.

Another important aspect of the distinction between symmetric and antisymmetric lies in voting/decision making. A voter can take an antisymmetric view with respect to candidates in a list of options (voting for or against a candidate) or a symmetric view (actively or passively abstaining). Sometimes a candidate that attracts the most votes is actually objectionable to a majority of voters, or all candidates are actually objectionable to a majority of voters.

Martin Hay has prototype code written for two social systems based on treating the components of the system as elements of unitary quaternions. One is for exchange/taxation, based on a chiral tetrahedron composed of the ligands me, you, mine and yours arranged around goods or services.



The other is a voting system based on a chiral tetrahedron composed of the ligands accept, reject, for and against arranged around candidates or options.

We hypothesize that brain chiral quaternion regulation of internal and changing ownership statuses among the brain's constituent operand elements, muscle or memory, body or mind, is a strategy by which the brain controls its resources and movements so as to keep everything in overall balance. It may also play a role in perception and decision making, with neurons acting as voters simultaneously responding to options from antisymmetric and symmetric viewpoints leading to a conclusion that emerges into consciousness.

The Chiral Tetrahedral Model and the Necker Cube

Overview

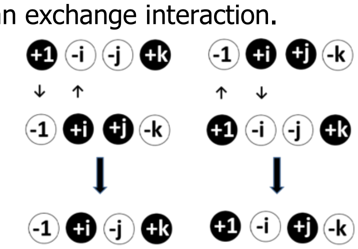
Our model enables information about relationships to be coded and manipulated as a chiral system of four ordered polarities that conform to quaternion multiplication rules. It has developed out of a study of accounting, chirality in tetrahedral molecules, and neuroscience.

We want to encourage use of the model to develop and test hypotheses about aspects of how the brain works, for example, to co-ordinate movement and to control aspects of perception.

Our model's coding is different from current-day systems used to code economic relationships (exchange, taxation and voting), which are based on two ordered polarities. Our code may therefore reveal insights into new and potentially better ways to code such relationships. If the code does indeed accord with how the brain works, then it may prove useful in the development of new treatments for disorders of the brain and also for the design of new kinds of computer/brain interface.

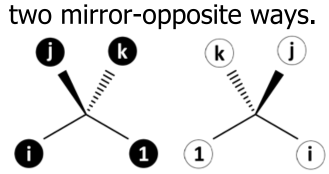
Chiral Tetrahedral Model

This model codes information about relationships as a co-ordinated system based upon four ordered polarities that function as chiral unitary quaternions. The system is controlled by complementary switching of polarities in two quaternions, as if the two quaternions have undergone an exchange interaction.



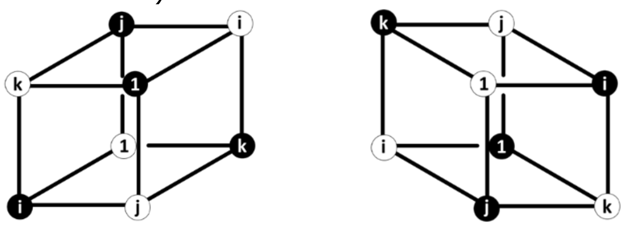
This section will derive the model with reference to chiral tetrahedral molecules and the Necker cube effect (Necker, 1832).

In three dimensional space four different objects 1, i, j and k can be arranged at the corners of a tetrahedron in two mirror-opposite ways.



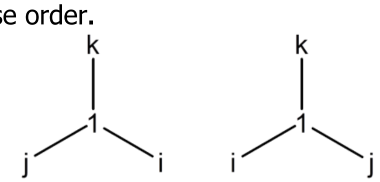
The two arrangements are related as the left and right hands. The tetrahedron is said to be chiral, after the Greek work for hand. The two mirror-opposite forms of the tetrahedron are called enantiomers of one another.

If the two enantiomers are superposed, the eight different objects form eight different corners of a cube. The cube has six different faces. It is chiral. The chiral cube has an enantiomer. Switching between the enantiomers corresponds with exchanging black and white. Each cube embodies what it is and what it is not (its enantiomer).



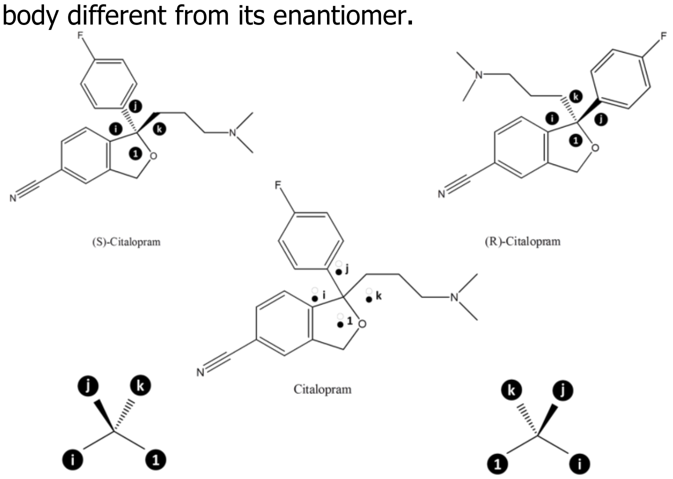
When the chiral cube is rotated about an axis the relative positions of the eight different corners and six different faces are conserved. The corners or faces on the rotation axis remain fixed while those off it move in unison.

The distinction between the two enantiomers arises, because i, j and k can be read in clockwise or anticlockwise order.



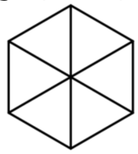
This order, or spin, is conserved as the chiral tetrahedron or chiral cube is rotated. Each enantiomer is characterised by the order.

Chirality is a fundamental symmetry of nature. It can be found at all scales, from fundamental particles to spiral galaxies. Nature respects chirality. Life is coded in chiral molecules. A medicine of one chiral form (e.g. the antidepressant (S)-citalopram) produces an effect on the body different from its enantiomer.

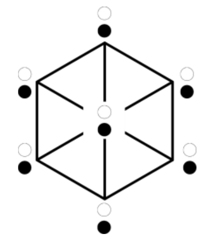


The phenomenon of chirality can be experienced through the Necker cube effect.

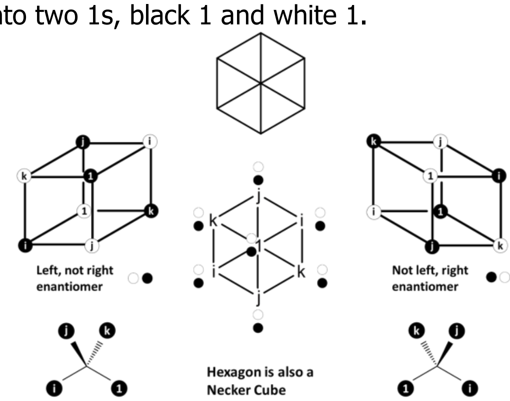
Consider this hexagon, a flat, symmetrical object.



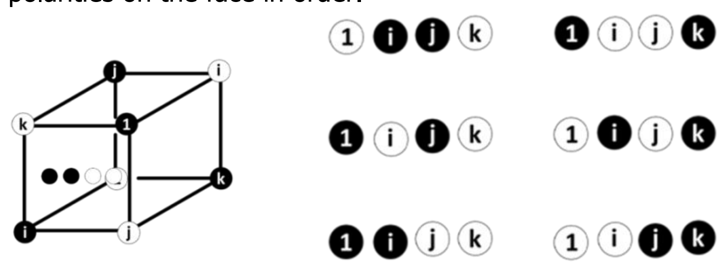
If you look at it carefully, a cube can appear to jump out. Many thanks to John Gaboury for pointing this out. The 2D figure is ambiguous in 3D, because the central point can form the top right front or bottom left back of a cube. Each of the corners can have two possible relative positions.



If the centre of the cube is labelled 1 and opposed corners are labelled i, j or k, then two cubes can jump out. The symmetry of the hexagon is broken as the 1 is split into two 1s, black 1 and white 1.



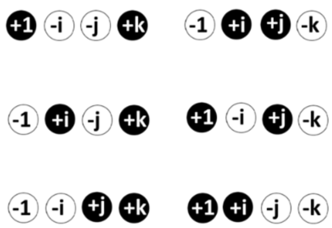
We code each face of the cube by arranging the polarities on the face in order.

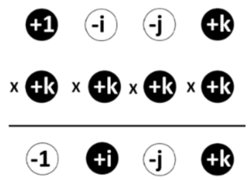


We need to relate these to the quaternion multiplication rules.

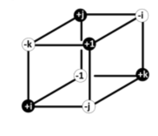
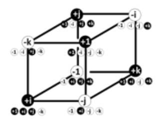
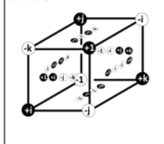
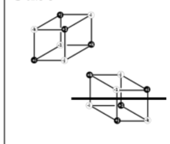
$$ij = k, \quad ji = -k,$$
$$jk = i, \quad kj = -i,$$
$$ki = j, \quad ik = -j$$
$$i^2 = j^2 = k^2 = -1.$$

In order to do this, we assign a positive sign to the black numbers and a negative sign to the white numbers. We can now effect rotation of the cube from one face to another by multiplying each polarity by the same quaternion. After constructing a multiplication table for all possible combinations, we find that each face behaves as a unitary quaternion.

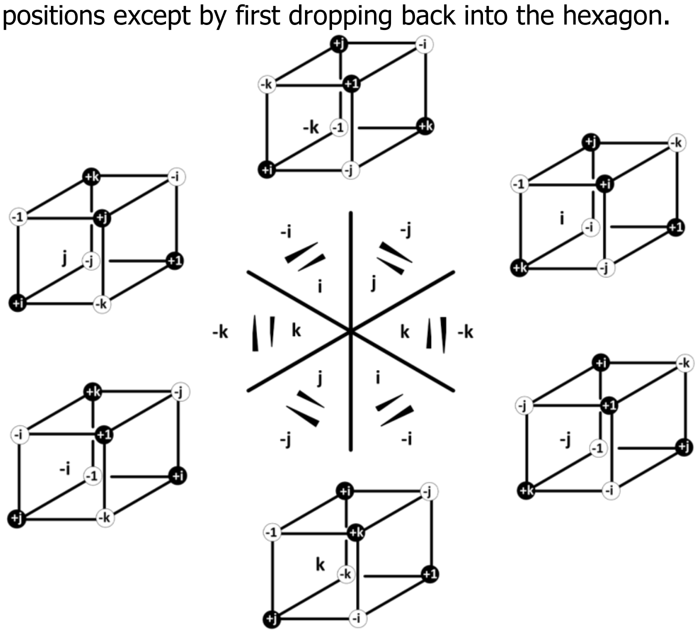




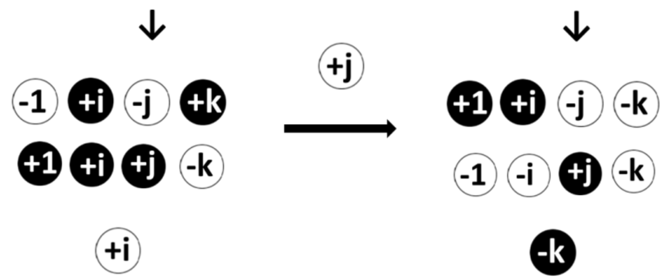
Indeed continuing the analysis, it turns out that each face and corner of the cube is coded as a unitary quaternion.

Quaternion	Corner	Face	Cube
			
+1 +1, 0i, 0j, 0k	+1 -i -j -k		+1 +i +j +k
-1 -1, 0i, 0j, 0k	-1 +i +j +k		-1 -i -j -k
+i 0, +i, 0j, 0k	+1 +i +j -k	-1 +i -j +k	
-i 0, -i, 0j, 0k	-1 -i -j +k	+1 -i +j -k	
+j 0, 0i, +j, 0k	+1 -i +j +k	-1 +i +j -k	
-j 0, 0i, -j, 0k	-1 +i -j -k	+1 -i -j +k	
+k 0, 0i, 0j, +k	+1 +i -j +k	-1 -i +j +k	
-k 0, 0i, 0j, -k	-1 -i +j -k	+1 +i -j -k	

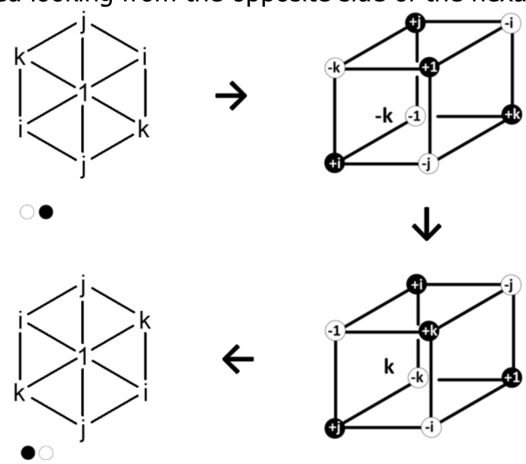
After a cube has jumped out of the hexagon, it is constrained by the quaternion multiplication rules to rotate between six positions about specific axes. It cannot flip into the other chirality or any other set of six positions except by first dropping back into the hexagon.



The rotation axis is controlled by the coding for the corners and faces. The axis is defined by the polarities that do not change in all four codes.

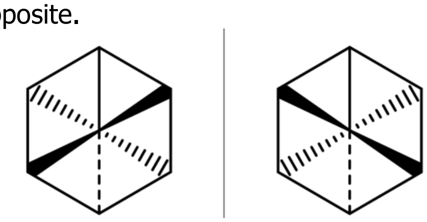


Two quaternions of the same kind, but of opposite polarity have all of their faces opposed. Otherwise none of the faces are opposed. The cube can only collapse down to the original hexagon if it is in the original, non-rotated position. However, in the special case when a quaternion has been rotated around to its signed opposite, it can collapse down to the hexagon with the order of i, j and k reversed: as if the cube had been created looking from the opposite side of the hexagon.



It follows that if cubes are transformed in pairs, through exchange interactions, then they can only collapse down to hexagons when the cubes are of the same kind, but opposite polarity. Otherwise they are *entangled* (bound, trapped).

The Necker cube effect breaks the symmetry of the hexagon. The three equivalent pairs of lines meeting at the centre become two pairs of one kind and one pair of two kinds. The one pair of two kinds is determined by the axis about which the quaternion can rotate into its signed opposite.



The members of the one pair of two kinds are related as one enantiomer is to another, or as i, j, k read clockwise is to i, j, k read anticlockwise. Each is one, not the other. This is the fundamental distinction upon which the whole system is constructed. Four such distinctions are arranged on order, and with every transformation that order is conserved. Every unitary quaternion, every chiral cube, every face and every corner has a dual that is related as one enantiomer to another. By coding relationships in pairs of unitary quaternions that consist of four ordered polarities, control can be asserted by treating all transformations as exchange interactions equivalent to performing complementary quaternion multiplication steps. All information about relationship is conserved. Prototypes for the exchange of goods and services without money and for voting/decision making have been constructed based on the system. The system may also be of interest for modelling the co-ordination of skeletal muscle antagonist pairs.

Results: What We Learned from Our Exploration of Our Model

In a sense, each reader of this paper is performing an experiment, through experiencing the Necker cube effect and relating it to unitary quaternions. There will no doubt be more to explain, and indeed further experimentation may show that the apparent relationship is mere co-incidence. Experiments to falsify the model need to be devised. Nevertheless, the model does appear to work as a mathematical system and to resonate with the work of other researchers in diverse fields (Schmeikal 2014; Gaboury, 2013; Pastukhov and Jochen, 2012; Goertzel, 2007)

We would like to reach out to other researchers to share our model and the tools we have built and explore how these could assist them in their work. In particular, we have learned to appreciate the phenomenon of the complementary *openings and closings of polarities* in the model – of how one form of distinction (polarity) can close down and disappear, and a new one rise and take its place. This lends itself well to psychology applications in which new viewpoints are emerging and begin to be explored by the mind/brain. This resonates well with Goertzel's model of working memory as an octonion- and quaternion -driven switching mechanism for bringing new entities and persona into consciousness.

Conclusions

Chirality is a symmetry that emerges in abstract spaces. It manifests on all scales of nature and is inextricably bound in the chemistry of living systems. It finds a natural mathematical representation in unitary quaternions.

Unitary quaternions may prove to be useful tools in neuroscience for modelling perception and control, and in the social sciences for coding information about relationships in social and economic systems.

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